Then $\|D^2u\|^2_{L^2(\Omega')} \le C(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \underline{\|Du\|^2_{L^2(\Omega)}})$

$\le \tilde{C}(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})$. $\quad \le C(\|f\|^2_{L^2} + \|u\|^2_{L^2})$

How about $u \in H^3(\Omega')$? $\quad Lu = D_i(a_{ij}D_ju) + b_iD_iu + cu = f, \quad (Lu)_k = f_k$.

$\Rightarrow D_i(a_{ij}D_j(u_k)) + D_i((a_{ij})_kD_ju) + b_iD_iu_k + (b_i)_kD_iu + cu_k + c_ku = f_k$.

Let $v = u_k$. $\Rightarrow D_i(a_{ij}D_jv) + b_iD_iv + cv = f_k - c_ku - (b_i)_kD_iu - D_i((a_{ij})_kD_ju)$

$\tilde{f} \equiv$ (the right-hand side above), where $c_ku \in L^2$, $(b_i)_kD_iu \in L^2$, and $D_i((a_{ij})_kD_ju) = (a_{ij})_kD_{ij}u + (a_{ij})_{ki}D_ju$ with $(a_{ij})_kD_{ij}u \in L^2(\Omega'')$, $(a_{ij})_{ki}D_ju \in L^2$.

Let's assume $|D^2a_{ij}| + |Db| + |Dc| \le C$,

and $Df \in L^2$. $\tilde{f} \in L^2(\Omega'')$. Apply the previous lemma, where $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.

$\|Du\|_{H^2(\Omega')} = \|v\|_{H^2(\Omega')} \le C(\|\tilde{f}\|_{L^2(\Omega'')} + \|v\|_{L^2(\Omega'')})$

$\le C(\|f\|_{L^2(\Omega'')} + \|Df\|_{L^2(\Omega'')} + \|u\|_{L^2(\Omega'')} + \|Du\|_{L^2(\Omega'')} + \|D^2u\|_{L^2(\Omega'')})$

$\le \tilde{C}(\|f\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)})$

$\Rightarrow \|u\|_{H^3(\Omega')} \le C(\|f\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)})$.

To get the estimation above, we simply take the derivative of the equation and apply the interior H^2-estimate. By taking 2^{nd} derivative of the eq and applying the interior H^2-estimate, we get

$\|u\|_{H^4(\Omega')} \le C(\|f\|_{H^2(\Omega)} + \|u\|_{L^2})$

$\vdots$

$\|u\|_{H^{2+k}(\Omega')} \le C(\|f\|_{H^k(\Omega)} + \|u\|_{L^2})$ if $\|a_{ij}\|_{C^{1+k}(\Omega)} + \|b_i\|_{C^k(\Omega)} + \|c\|_{C^k(\Omega)} < +\infty$.

We call them boot step argument.

5/26 월

Let's consider simple case

$Lu = -D_i(a_{ij}(x)D_ju) = f$.

$B[u,\varphi] = \int_\Omega a_{ij}(x)D_iuD_j\varphi\,dx = \int_\Omega f\varphi\,dx$.

Let $\varphi = D_k(\eta^2D_ku)$.

$\int_\Omega a_{ij}(x)D_iuD_jD_k(\eta^2D_ku) = \int_\Omega fD_k(\eta^2D_ku)$.

$= -\int_\Omega D_k(a_{ij}(x)D_iu)D_j(\eta^2D_ku) = \int_\Omega f(2\eta\eta_kD_ku + \eta^2D_{kk}u)$

$-\int_\Omega (\underline{a_{ij}D_{ik}u} + D_ka_{ij}D_iu)(\eta^2\underline{D_jD_ku} + 2\eta D_j\eta D_ku)$

$\lambda\int_\Omega \eta^2|D^2u| = \lambda\sum_k\int_\Omega \eta^2|Du_k|^2 \le \sum_k\int_\Omega \eta^2a_{ij}D_iu_kD_ju_k$

$\le C\sum_k\int_\Omega \eta^2|Da||Du||Du_k| + \eta|D\eta||Du|^2 + |f||\eta||D\eta| + |\eta^2||f||D_{kk}u|$

$\lambda/4C \qquad \lambda/4C$

$$\le \frac{\lambda}{2}\int_\Omega \eta^2|D^2u| + \frac{\tilde{c}}{\lambda}\int_\Omega |Du|^2+|f|^2,$$

$$\frac{\lambda}{2}\int_\Omega \eta^2|D^2u| \le \frac{\tilde{c}}{\lambda}\int_\Omega |Du|^2+|f|^2 \qquad \eta\equiv 1 \text{ on } \Omega'$$

$$\Rightarrow \int_{\Omega'}|Du|^2 \le \int_\Omega \eta^2|Du|^2 \le C\int_\Omega |Du|^2+|f|^2.$$

The computation is ok if $u\in H^3$. We just know $u\in H^1$

$D_i^h u = \dfrac{u(x+he_i)-u(x)}{h}$: Differential quotient.

$a^h(x) = a(x+h)$.

- $D_k^h(uv) = u^h D_k^h v + (D_k^h u)v \in H^1$
- $\int_\Omega u(x) D_k^h v = -\int_\Omega (D_k^{-h}u)v$ if $\operatorname{supp} v \subset\subset \Omega$ and h is small.

$\varphi = D_k^{-h}(\eta^2 D_k^h u) \in H^1$ for any $h>0$,

$$\int_\Omega a_{ij}(x) D_i u D_j D_k^{-h}(\eta^2 D_k^h u) = \int_\Omega f D_k^{-h}(\eta^2 D_k^h u)$$

$$= -\int_\Omega D_k^{+h}(a_{ij}D_i u) D_j(\eta^2 D_k^h u) = \int_\Omega f\,(2\eta^{-h}D_k^h\eta D_k^h u + (\eta^2)^{-h}D_{kk}u)$$

$$-\int_\Omega (a_{ij}^h D_{ik}u + D_k^h a_{ij} D_i u)(\eta^2 D_j D_k^h u + 2\eta D_j\eta D_k^h u)$$

$$\lambda\int_\Omega \eta^2|DD_k^h u| = \lambda\sum_k\int_\Omega \eta^2|DD_k^h u|^2 \le \sum_k\int_\Omega \eta^2 a_{ij} D_i D_k^h u D_j D_k^h u$$

$$\le C\sum_k\int_\Omega \eta^2|Da|_{L^\infty}|Du||DD_k^h u| + \eta|D\eta||Du|^2 + |f||\eta||D\eta| + |\eta|^2|f||D_{kk}^h u|$$

$$\le \frac{\lambda}{2}\int_\Omega \eta^2|DD_k^h u|^2 + \frac{\tilde{c}}{\lambda}\int_\Omega |Du|^2+|f|^2$$

Since $\int_\Omega |D_k^h u|^p dx \le C\int_\Omega |Du|^p dx$ for $0<p<+\infty$

$$\int_{\Omega'} |DD_k^h u|^2 dx \le C\int_\Omega |Du|^2+|f|^2 \le \underline{C}(\|f\|_{L^2}^2 + \|u\|_{L^2(\Omega)}^2) \quad \text{for any } 0<h\ll 1.$$

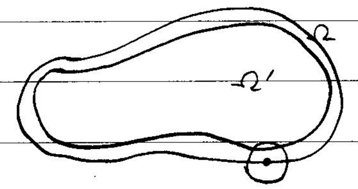
Let $h\to 0$. Then

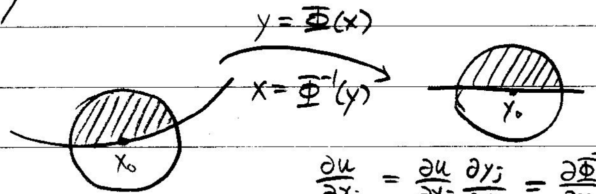
$$\int_{\Omega'}|D^2u|^2 dx \le C(\|f\|_{L^2}^2 + \|u\|_{L^2}^2)$$

Since C is indep. of h.

(여기 숙제 HW1 이 있음. 뒤에 따로 표시)

⟨Boundary Regularity⟩





$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial y_j}\frac{\partial y_j}{\partial x_i} = \frac{\partial \Phi^{-1}}{\partial x_i} D_{y_j} u.$$

$$\begin{cases} Lu = f(x) & \text{in } \Omega\cap B_\rho(x_0) \\ u=\varphi & \text{on } \partial\Omega\cap B_\rho(x_0)\end{cases} \longrightarrow \begin{cases} \bar{L}\bar{u} = \bar{f}(x) & \text{in } \mathbb{R}_n^+\cap B_{\bar\rho}(y_0) \\ \bar{u} = \bar\varphi & \text{on } \{x_n=0\}\cap B_{\rho'}(y_0)\end{cases}$$

where $\bar f(y) = f(x(y))$

$\bar\varphi(y) = \varphi(x(y)) = \varphi(\Phi^{-1}(y))$

$B[u,\varphi] = \int_\Omega a_{ij}(x) D_{x_i}u D_{x_j}\varphi\, dx \qquad (Lu = -D_i(a_{ij}(x)D_j u))$

$= \int_{\mathbb{R}^n_+ \cap \Phi(\Omega)} a_{ij}(\Phi^{-1}(y)) \frac{\partial \Phi^k}{\partial x^i} \frac{\partial \Phi^l}{\partial x^j} D_{y_k} u \, D_{y_j} \varphi \det\left(\frac{\partial \Phi^k}{\partial y^l}\right) dy$

$= \int_{\mathbb{R}^n_+ \cap \Phi(\Omega)} \bar{a}_{ij}(y) D_{y_k} u \, D_{y_l} u \, dy$

where $\bar{a}_{ij}(y) = a_{ij}(\Phi^{-1}(y)) \frac{\partial \Phi^k}{\partial x^i} \frac{\partial \Phi^l}{\partial x^j} \det\left(\frac{\partial^l \Phi}{\partial y^k}\right) < +\infty.$

Check : $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \Rightarrow \bar{\lambda}|\xi|^2 \le \bar{a}_{ij}\xi_i\xi_j \le \bar{\Lambda}|\xi|^2$ where $\bar{\lambda}, \bar{\Lambda}$ depends on the Lipschitz norm of Φ.

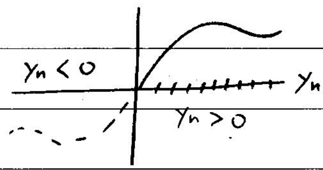
I.e. if a_{ij} is uniformly elliptic and if Φ is Lipschitz, then $\bar{a}_{ij}(y)$ is also uniformly elliptic.

$Lu = -\Delta u$ → $\bar{L}u = -D_i(\bar{a}_{ij}(y) D_j u)$

- If we expect $\bar{a}_{ij}(y) \in Lip$ for H^2 regularity, we need the conditions :

  $a_{ij} \in Lip$ and $D\Phi \in Lip$ (or $\partial\Omega \in C^2$)

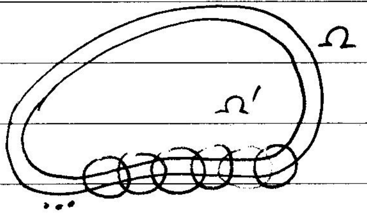
And $u \in H^1(\mathbb{R}^n_+)$ and $u = 0$ on $y_n = 0$ $\Rightarrow$ the odd extension of u ($u(x', -x_n) = -u(x', x_n)$) belongs to $H^1(\mathbb{R}^n)$.



Let $\bar{u}(x) = \begin{cases} u(x', x_n) & \text{if } x_n > 0 \\ -u(x', -x_n) & \text{if } x_n < 0 \end{cases}$ satisfies $\{ L\bar{u} = f$ in $B_\rho(y_0)$, then

the interior H^2- estimate says $\|\bar{u}\|_{H^2(B_{\rho/2}(y_0))} \le C(\|\bar{u}\|_{L^2(B_\rho(y_0))} + \|\bar{f}\|_{L^2(B_\rho(y_0))})$

$= 2C\, \|\bar{u}\|_{L^2(B^+_\rho(y_0))} + \|\bar{f}\|_{L^2(B^+_\rho(y_0))})$



Since $\overline{\Omega \setminus \Omega'}$ is compact, we just need finite nb of balls to cover it. And then add the inequalities to get

$\|u\|_{H^2(\Omega)} \le \|u\|_{H^2(\Omega')} + \sum_{k=1}^{m} \|u\|_{H^2(\Omega \cap B_\rho(x_k))}$

(↑ interior estimate; ↑ boundary estimate)

$\le C(m_2)(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).$

Thm (Global estimate)

$|Da|_{L^\infty(\Omega)} + |b|_{L^\infty(\Omega)} + |c|_{L^\infty(\Omega)} \le M.$

The weak sol u of $\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ belongs to $H^2(\bar{\Omega}) \cap H^1_0(\Omega)$ and

$\|u\|_{H^2(\Omega)} \le C(\|u\|_{L^2} + \|f\|_{L^2}).$

⟨Eigen Value Problem⟩

$$Lu = -D_i(a_{ij} D_j u)$$

Find (λ, u) s.t. $\begin{cases} Lu = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$, $B(u,u) = \int_\Omega a_{ij}(x) D_i u D_j u$.

Consider the following Rayleight's quotient:

$$E(u) = \frac{B(u,u)}{\|u\|_{L^2}^2} = \frac{\int_\Omega a_{ij}(x) D_i u D_j u \, dx}{\int_\Omega |u|^2 dx}$$

Set $\lambda_1 = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} E(u)$.

(i) If u_1 is the enegyminizer of $E(u)$, then u_1 is the eigen ft corresponding to an eigenvalue λ_1.

(ii) λ_1 is the smallest eigenvalue.

(iii) $u_1 > 0$ or $u_1 < 0$ in Ω.

(iv) Let $H_1 = \{u \mid Lu = \lambda_1 u\}$, $\dim H_1 = 1$.

(v) There is such u_1 (Existence).

Proof) (i) For any $v \in H_0^1(\Omega)$ and $u + tv \neq 0$, $E(u) \le E(u+tv) = \frac{B(u+tv, u+tv)}{(u+tv, u+tv)} = g(t)$.

$g'(0) = 0$, $\qquad \frac{B(u,u) + 2tB(u,v) + t^2 B(v,v)}{(u,u) + 2t(u,v) + t^2(v,v)}$

$$g'(t) = \frac{2B(u,v) + 2tB(v,v)}{(u,u) + 2t(u,v) + t^2(v,v)} - \frac{(\quad)}{[\quad]^2}(2(u,v) + 2t(v,v))\Big|_{t=0}$$

$$= \frac{2B(u,v)}{(u,u)} - 2\frac{B(u,v)}{(u,u)^2}(u,v) = 0.$$

$\Rightarrow B(u,v) = \frac{B(u,u)}{(u,u)}(u,v) = (\lambda_1 u, v)$ since $\lambda_1 = \frac{B(u,u)}{(u,u)}$. for all v.

$\Rightarrow Lu = \lambda_1 u$ in Ω.

(ii) If $\exists u \neq 0$ s.t. $Lu = \lambda u$, $B(u,v) = (\lambda u, v)$. Let $v = u$,

$$\lambda = \frac{B(u,u)}{(u,u)} \ge \lambda_1.$$

(iii) $E(u) = \frac{\int_\Omega a_{ij} D_i u D_j u}{\int_\Omega |u|^2 dx} = \frac{\int_\Omega a_{ij} D_i |u| D_j |u|}{\int_\Omega ||u||^2 dx} = E(|u|)$.

If u is minimizer of $E(u)$, then so is $|u|$. So are $u^+ = \frac{u + |u|}{2}$, $u^- = \frac{|u| - u}{2}$.

Then $\begin{cases} Lu^+ = \lambda_1 u^+ & \text{in } \Omega \\ u^+ = 0 & \text{on } \partial\Omega \\ u^+ \ge 0 \end{cases}$. The strong minimum principle says $u^+ > 0$ in Ω or $u^+ \equiv 0$ in Ω,

$\Rightarrow u > 0$ in Ω or $u < 0$ in Ω.

(iv) If $\dim H_1 = 2$, $\exists u_1 \neq 0, u_2 \neq 0$ s.t. $(u_1, u_2)_{L^2} = 0$.

From (iii), $\begin{cases} u_1 > 0,\ u_2 > 0 \Rightarrow (u_1, u_2) > 0 \\ u_1 > 0,\ u_2 < 0 \Rightarrow (u_1, u_2) < 0 \\ u_1 < 0,\ u_2 > 0 \Rightarrow (u_1, u_2) < 0 \\ u_1 < 0,\ u_2 < 0 \Rightarrow (u_1, u_2) > 0. \end{cases}$

(v) $E(u) = \frac{B(u,u)}{\|u\|_{L^2}^2} = B\left(\frac{u}{\|u\|_{L^2}}, \frac{u}{\|u\|_{L^2}}\right) > 0.$ for $u \in H_0^1(\Omega)$, $u \neq 0$.

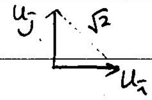
There is a minizing sequence s.t. $E(u_\lambda) \searrow \lambda_1$. We may assume $\|u_\lambda\|_{L^2} = 1$.

Then $B(u_\lambda, u_\lambda) \to \lambda_1$. We need some estimate to show $u_\lambda \to u \in H^1$.

$\lambda \|Du\|_{L^2}^2 \leq \underline{B(u,u)} \leq \Lambda \|Du\|_{L^2}^2$.

$B(u_\lambda - u_j, u_\lambda - u_j) + B(u_\lambda + u_j, u_\lambda + u_j) = 2\left(B(u_\lambda, u_\lambda) + B(u_j, u_j)\right)$

$$\begin{aligned}\lambda \|D(u_\lambda - u_j)\|_{L^2}^2 &\leq B(u_\lambda - u_j, u_\lambda - u_j) \\ &= 2\left(B(u_\lambda, u_\lambda) + B(u_j, u_j)\right) - \frac{B(u_\lambda + u_j, u_\lambda + u_j)}{\|u_\lambda + u_j\|^2}\left(\|u_\lambda + u_j\|^2\right) \\ &\leq 2\left(B(u_\lambda, u_j) + B(u_j, u_j)\right) - \lambda_1\left(\underbrace{\|u_\lambda + u_j\|^2}_{=2}\right) \\ &\searrow 2(\lambda_1 + \lambda_1) - 2\lambda_1 = 0.\end{aligned}$$



So $\{u_\lambda\}$ is Cauchy seq. in $H_0^1(\Omega)$.

$\exists u \in H_0^1(\Omega)$ s.t. $u_\lambda \to u_1$ in $H_0^1(\Omega)$. $\|u_1\|_{L^2} = 1$.

$E(u_1) = \lim_{\lambda} E(u_\lambda) = \lambda_1$. ▨

HW. p347 # 9, 10.

HW1 (이전 표시한 곳에 있어야 할 숙제)

(i) If $u \in W^{1,p}(\Omega)$, $1 < p < \infty$, then $\|D_k^h u\|_{L^p} \leq C\|Du\|_{L^p}$

(ii) If $\|D_k^h u\|_{L^p} \leq C$ uniformly for any $0 < h \ll 1$, $u \in W^{1,p}$ and $\|Du\|_{L^p} \leq C$.